Eur. Phys. J. C 14, 705–708 (2000)
Digital Object Identifier (DOI) 10.1007/s100520000375

THE EUROPEAN PHYSICAL JOURNAL C

Vector fields, flows and Lie groups of diffeomorphisms

A. Peterman

Theoretical Physics Division, CERN, 1211 Geneva 23, Switzerland

Received: 25 February 2000 / Published online: 18 May 2000 – © Springer-Verlag 2000

To the memory of G. de Rham, my teacher in mathematics.

Abstract. The freedom in choosing finite renormalizations in quantum field theories (QFT) is characterized by a set of parameters $\{c_i\}, i = 1 \ldots, n \ldots$, which specify the renormalization prescriptions used for the calculation of physical quantities. For the sake of simplicity, the case of a single c is selected and chosen mass-independent if masslessness is not realized, this with the aim of expressing the effect of an infinitesimal change in c on the computed observables. This change is found to be expressible in terms of an equation involving a vector field V on the action's space M (coordinates x). This equation is often referred to as "evolution equation" in physics. This vector field generates a one-parameter (here c) group of diffeomorphisms on M. Its flow $\sigma_c(x)$ can indeed be shown to satisfy the functional equation

$$\sigma_{c+t}(x) = \sigma_c(\sigma_t(x)) \equiv \sigma_c \circ \sigma_t$$

$$\sigma_0(x) = x,$$

so that the very appearance of V in the evolution equation implies at once the Gell-Mann-Low functional equation. The latter appears therefore as a trivial consequence of the existence of a vector field on the action's space of renormalized QFT.

The so-called "Renormalization Group" (RG) in physical science was, from its early beginnings [1], the theory that describes the geometry of action space. In this space the covariance of physical quantities turns out to be manifest.

This paper is intended to be an overview of the RG as it is used by physicists, especially in Quantum Field Theory (QFT). However, the emphasis will be put on the geometry (as said before) of the space of actions, with a view, among other things, on how a functional equation, derived in [2], becomes a simple and trivial consequence of the geometrical set-up developed below by means of QFT manipulations.

Of paramount importance has been the discovery [1] that renormalized physical quantities satisfy equations in which the basic geometrical object is a vector field defined in the action space M. Let V be this vector field. Then the theory of differentiable manifolds [3] implies, due to the very existence of a vector field, a set of theorems, lemmas and corollaries which exhausts all that can be said about the renormalization group in physical applications. In general, one deals with a set of parameters $\{c_i\}$ but, in the following, we shall restrict this set to a single parameter, denoted t for practical reasons. This restriction is mainly dictated by the fact that one wants to be able to make direct comparisons with [2] which uses a single parameter in its fixing of renormalization prescriptions. Thus, in the single parameter case, we have amongst others the following theorems.

Theorem I. A smooth vector field V on a compact manifold M generates a one-parameter group of diffeomorphisms of M.

Theorem II. Suppose V is a C^r vector field on the manifold M, then for every $x \in M$, there exists an integral curve of V, $t \to \sigma(t, x)$ such that

1. $\sigma(t, x)$ is defined for t belonging to an interval $I(x) c R$, containing $t = 0$ and is of class C^{r+1} there.
2. $\sigma(0, x) = x$ for every $x \in M$
3. Given $x \in M$, there is no C^1 integral curve of V defined on an interval properly containing $I(x)$, and passing through x (i.e. such that $\sigma(0, x) = x$).

From the uniqueness property 3, follows at once

Theorem III. If s, t and $s + t \in I(x)$, then we have the functional equation

$$\sigma(s + t, x) = \sigma(t, \sigma(s, x)) \,. \qquad (1)$$

Flow: The set of pairs $(x, t), x \in M, t \in I(x)$ is an open subset of $M \times R$ containing x, hence a smooth manifold $\sum$, of dimension $n + 1$. The mapping σ: $\sum_v \to X$ by $(x, t) \to \sigma(t, x)$ is called the flow of the C^1 vector field V.

If M and V are C^∞, the flow is also of class C^∞. Writing $\sigma(t,\cdot) \equiv \sigma_t : x \to \sigma(t,x)$; (1) can be written

$$\sigma_{s+t} = \sigma_t \circ \sigma_s \quad ; \quad \sigma_0 = e \; . \tag{2}$$

The set of mappings $\{\sigma_t\}$ is the one-parameter group mentioned by Theorem I, provided, as is the case for our concern, $I = R$. (It is obvious that each element σ_t has an inverse σ_{-t} so that $\sigma_t \circ \sigma_{-t} = e$)[1].

According to the few theorems given up to now, one sees that the vector field V is the *key* concept, generating a flow $\sigma(t,x) \equiv \sigma_t(x)$ in M. Since the σ_t, with fixed t, is a diffeomorphism $M \to M$, it represents the one-parameter Abelian group according to (2). Therefore, V can be seen as the infinitesimal generator of the flow group σ_t. Indeed, for t infinitesimal, say $0+\delta t$, we have the infinitesimal flow

$$\sigma_{0+\delta t}(x) = \sigma_0(x) + \delta t V(x) + O(\delta t^2)$$

(with $\sigma_0(x) = x$), by Taylor expending $\sigma(\delta t, x)$ around $\delta t = 0$, so that

$$\frac{\sigma(0+\delta t, x) - \sigma(0,x)}{\delta t} = V(x) \; .$$

Or, taking the limit $\delta t \to 0$

$$\left. \frac{d\sigma(t,x)}{dt} \right|_{t=0} = V(x) \; . \tag{3}$$

$V(x)$ is therefore, as we said at length before, the infinitesimal generator of the flow group. Then, by exponentiation, one gets

$$\sigma(t,x) = \exp[tV] \cdot x$$

which fulfils, as expected

a) $\sigma(0,x) = x$
b) $\frac{d}{dt}\sigma(t,x) = V \exp[tV] \cdot x = V(\sigma(t,x))$
c) $\sigma(s+t,x) = \sigma(t, \sigma(s,x))$

(Remember the operator nature of V, which can be abbreviated $V = V^\alpha \frac{\partial}{\partial x^\alpha}$.)

The elementary considerations made up to now, would be sufficient for the current applications of the RG to QFT since, as is well-known, the one-parameter t is the logarithm of a scale μ, and $\frac{d}{dt} \to \mu \frac{d}{d\mu}$, which is the Abelian generator of the one-parameter group for scale transformations of the subtraction point.

The appearance of the arbitrary dimensional parameter μ^2 cannot be avoided and is the source of the breakdown of conformal and scale invariances in classical conformal invariant Lagrangians. The deep source of this anomalous breakdown is traced back, as is well-known, in the procedure of the second quantization [4]. In this reference, R. Jackiw notices " … we may say that our present point of view towards scale and conformal symmetry breaking was prefigured by Bohr's intuition concerning effects of quantization on space-time symmetries."

In conclusion, in order to fix the ideas in the present "physical" notations, V is expressed as

$$V = \beta^a(g) \cdot \frac{\partial}{\partial g^a} \quad ; \quad a = 1 \ldots \kappa \; ,$$

if there are κ couplings in the considered theory. $g \equiv \{g^a\}$ is the set of couplings and $\{\beta^a(g)\}$ that of the components of the vector field.

We see therefore that the β^a are the components V^α of V and the g^a, the coordinates x^α of M (called in physics the action space). Finally, the components of the flow in this space are denoted $\bar{g}^a(t,g)$, corresponding to the $\sigma^\alpha(t,x)$ components of the flow $\sigma(t,x)$ on M.

So that, since, as we have seen, V is the infinitesimal generator of the one-parameter flow group on M, $\beta^a \frac{\partial}{\partial g^a}$ is the one-parameter flow group infinitesimal generator on action space. As a flow, $\bar{g}(t,g)$ can be obtained from the exponentiation of $V = \beta^a \frac{\partial}{\partial g^a}$ (see the example in the Appendix A). It therefore satisfies the functional equation (1) as expected, namely:

$$\bar{g}(t+s, g) = \bar{g}(t, \bar{g}(s,g))$$

with $\bar{g}(0,g) = g$ as boundary condition[3].

The fundamental equation for QED S-matrix elements, already quoted in its simplest formulation in the Abstract of [1], and which has been mentioned in this paper as the equation introducing in physics a vector field $\underline{V}$ was

$$\left. \frac{\partial}{\partial c_i} S(x \ldots m, e, c_i) \right|_{c_i=0} = h_{i.e.}(e) \frac{\partial}{\partial e} S(x \ldots, m, e) \tag{4}$$

or, in the simplified case (one single c: $c_0 = t$) of this paper

$$\left. \frac{\partial}{\partial t} S(p_j \ldots m, e, t) \right|_{t=0} = \underline{V} \cdot S(p_j \ldots m, e) \tag{5}$$

(the p_j being a conjugate momenta of the x'_s in (4)), with of course $\underline{V} = h_0(e) \frac{\partial}{\partial e}$.

Notice that the index i has been dropped in the formulation of this paper, since it stands for the numbering of the various arbitrary normalization conditions.

To our knowledge, the first author who took into consideration the general case [1] with several parameters c_i is Crewther [5]. His analysis is confined to a finite set of normalization conditions $R(c_i)$, and he put forward the very simple argument that it is sufficient to consider transformations in the c_i-space possessing the group property, which warrant a satisfactory rule for this special subset $R(c_i)$ of normalization conditions to have the group property. This is what was called "normalization group" in [1].

[1] For a sample of textbook' references involving the fundamentals of differential geometry as well as rigorous proofs of the topics advocated here and beyond these topics, see the item: *Textbooks* in the references at the end of the paper

[2] In physics, μ is generally called "subtraction point"

[3] In [2], $\bar{g}(t,g)$ is expressed as $e^2 d(t, e^2)$, with $t = \log(\kappa^2/\lambda^2) \equiv \log x^2$ in their notations. e^2 stands for g up to a change of coordinate since the QED case is investigated, with one *single* parameter λ, as in the formulation of the present paper

So the group property is a feature of many subsets of the whole set of prescriptions, G, but certainly G itself (the countable infinite set of prescriptions) does not, strictly speaking, possess this property.

A very popular set of prescriptions are the so-called "mass independent schemes", for which the normalization factors are computed with the bare mass set equal to zero. Then the renormalized mass is treated like a renormalized coupling [6]. In accordance with naïve dimensional analysis, the vector field components can only depend on the couplings [7]. In this set one finds, among others, dimensional regularization supplemented by minimal subtraction (MS) or its cousin $\overline{\mathrm{MS}}$. This mass-independent set of prescriptions possesses the group property. Several authors [8] tried to tackle the case when the c_i are infinite in number, especially with the aim of optimizing the perturbation series truncated at a given order. Although the possible group property in these extreme cases has not been addressed, these authors established that the most general coupling constant $g(c_1, c_2, \dots c_\infty)$ depends, as we wrote, on the countable infinity of c_i and were able to show in a particular case that the c_i are linear in the b_i, the numerical coefficients in the expansion in g of the vector field component $\beta(g) = \sum_{i=n}^{\infty} b_i g^i, n = 1, 2 \dots$. These coefficients are well-known for their dependence on the prescription used. In other words, geometrically, they depend on the choice of the coordinate $\{g^a\}$ in action space. However, the choice of a system of reference is arbitrary and the above results do not shed light on which sets, if any, enjoy the group property in a Banach space.

In conclusion, the passage from a single parameter $c_0 \equiv t$ considered in this paper, to several c_i, or an infinity of them, is *not* straightforward. Again we might be able to consider, as was done in [1], sets of transformations in the c_i space which possess the group property. This problem involves the theory of several parameter Lie groups of transformations and lies beyond the modest scope of this paper. Nevertheless, a few guidelines will be given in Appendix B in a very concise and not mathematically rigorous way.

Appendix A

As a very simple example, we take, for the vector field $V \Longleftrightarrow \beta(g)\frac{\partial}{\partial g}$ in a one-dimensional action space (coordinate g), the first term of $\beta(g)$ in a g expansion, say

$$\beta(g) = bg^2 .$$

The exponential $\exp\{t\underline{V}\} \cdot x$ is defined by its Taylor expansion

$$\exp\{t\underline{V}\} = \sum_{n=0}^{\infty} (t\underline{V})^n \frac{1}{n!}$$

From

$$\underline{V}\,\underline{V} \to bg^2 \frac{\partial}{\partial g}\left(bg^2 \frac{\partial}{\partial g}\right) = 2b^2 g^3 \frac{\partial}{\partial g} + b^2 g^4 \frac{\partial^2}{\partial g^2}$$

it is straightforward to deduce

$$\underbrace{\underline{V}\,\underline{V} \dots \underline{V}}_{n\,\text{factors}} = n! b^n g^{n+1} \frac{\partial}{\partial g} + O\left(\frac{\partial^n}{\partial g^n}\ ,\ n \geq 2\right) .$$

Therefore

$$\bar{g}(t, g) = \exp\{t\underline{V}\} g = \sum_{n=0}^{\infty} t^n b^n g^{n+1} = \frac{g}{1 - tbg} ,$$

a well-known result.

As an exercise, the reader can establish, according to the above, the approximate Bogoljubov-Shirkov relation in the next order for $\bar{g}(t, g)$ [9]

$$\bar{g}(t, g) = g[1 - b_1 gt + \frac{b_2}{b_1} g \log(1 - b_1 gt)]^{-1}$$

by taking

$$\beta(g) = b_1 g^2 + b_2 g^3 .$$

A second exercise is to show that

$$\exp\{t\underline{V}\} g^n = \bar{g}^n(t, g) = \frac{g^n}{(1 - gbt)^n}$$

when the vector field $\underline{V} = \beta(g)\frac{\partial}{\partial g}$ is approximated by the first term in the g expansion of $\beta(g)$ i.e. $\beta(g) = bg^2$, like in the first example.

Since S-matrix elements can be expanded in powers of g

$$S(p_i, g) = \sum_{n=0}^{\infty} a_n(p_i) g^n$$

it follows that

$$\exp\{t\underline{V}\} \cdot S(\dots g) = S(\dots \bar{g})$$

(p_i and the dots stand for arguments other than g and independent of it. The case when other couplings, like g_i and masses m_i occur, goes outside the one-dimensional action space and $\underline{V}$ becomes $V = V^\alpha \frac{\partial}{\partial x^\alpha}$, the x^α being the coordinates in the enlarged action space, namely the g_i and m_i above.)

Appendix B

The passage from a one-parameter case to the case with several parameters c_i is far from trivial, although treated at length in the Textbook references, especially [T.1]–[T.4]. Sketching what happens, from an element g depending on one parameter $g(t) \cdot g(s) = g(s+t)$ with $g(t) = 1 + t\underline{V}$ for t infinitesimal one goes to

$$g(t_1, \dots t_i) = 1 + t_1 \underline{V}_1 + t_2 \underline{V}_2 + \dots t_i \underline{V}_i$$

with all t_i infinitesimal. (i generators $\underline{V}_i$)

The combination of two such elements $g(t_1, \dots t_1) \cdot g(s_1, \dots s_i)$ is given by the well-known Baker-Campbell-Hausdorf formula.

For the product to be also an element of the set, the condition $[\underline{V}_i, \underline{V}_j] = c^{\kappa}_{ij}\underline{V}_{\kappa}$ is necessary and sufficient. It was a condition explicitly formulated in [1] for a set of normalization conditions to form a group. For our concern, the $g(t)$ are connected with the flow $\sigma_t(\cdot)$. Therefore the group property concerns the flows, as in the one-parameter case discussed in this paper.

The non-trivial aspect now is that we must distinguish between the left combination of $g(t_i)$ with $g(s_i)$ from the right combination. All this is treated in the mentioned textbooks and goes beyond the scope of the present modest account of vector fields.

References

1. E.C.G. Stueckelberg and A. Peterman, *Helv. Phys. Acta* **26** 1953 499
2. M. Gell-Mann and F.E. Low, *Phys. Rev.* **95** (1954) 1300
3. G. de Rham, Variétés différentielles, Actualités Scientifiques et Industrielles 1222 (Hermann, Paris, 1960)
4. R. Jackiw, Quantum mechanical symmetry breaking: recent developments in quantum field theory, Nils Bohr centennial conferences 1985, p. 203 (J. Ambjørn et al. eds. North-Holland, Amsterdam 1985.) See also, by the same author, his considerations on quantum symmetry breaking (in "Explorations in Theoretical Physics" December 1992, Cambridge, Mass. Also preprint CTP-2178, January 1993)
5. R. Crewther, Cargese Lectures (1975) (M. Lévy et al. eds. Plenum Press New York, p. 352)
6. K.E. Eriksson, *Nouv. Cim.* **30** (1963) 1423
7. S. Weinberg, *Phys. Rev.* **D8** (1973) 3497
8. P.M. Stevenson, *Phys. Rev.* **D23** (1981) 2916; G. Grunberg, *Phys. Rev* **D29** (984) 2315; S.J. Brodsky, G. Lepage and P. Mackenzie, *Phys. Rev.* **D28** (1983) 228
9. N. Bogoljubov and D. Shirkov, *Doklady. Akad. Nauk. SSSR* **103** (1955) p. 391–4

Textbooks. Rigorous proofs lie beyond the scope of this paper. They can be found

a) for mathematicians with a view on physics, for instance in Y. Choquet-Bruhet et al. [T.1], especially Chapter III, Sections A and B. Sections C and D offer a generalization to several-parameter Lie groups with vector fields $V_i, i = 1 \ldots n$, as infinitesimal generators, n = dim Lie Algebra = dim G. For $n = \infty$, see Chapter VII, Section A.

b) for mathematicians, in several treatises including this subject: for example K. Yano [T.2] Chapters I to VII included; S. Helgason [T.3], Chapters I and II (there our V is denoted by X, and our σ_t by $\gamma(t)$). Propositions 5.3 and theorem 6.1 of Chapter I are cornerstones to the rigorous proofs. Chapter II offers the generalization from one-parameter to several-parameter Lie group algebras (§1). See also S. Kobayashi and T. Nomizu [T.4].

c) For physicists, in oversimplified compendiums of differential geometry, like, for instance some chapters of [T.5], with definitions of the concepts used here and some sketches of proofs. A valuable reading is also A. Visconti [T.6].

[T.1] Y. Choquet-Bruhat, C. De Witt-Morette and M. Dillard-Bleick, Analysis, Manifolds and Physics, (North-Holland, Amsterdam, 1977 and subsequent editions, 1982 etc.).

[T.2] K. Yano, The theory of Lie derivatives and its applications, (North-Holland, Amsterdam, 1957).

[T.3] S. Helgason, Differential geometry, Lie groups and Symmetric spaces, (Academic Press, New York, 1978).

[T.4] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, I, II (Wiley Interscience, New York, 1963).

[T.5] R. Bertlmann, Anomalies in Quantum Field Theory (Clarendon Press, Oxford, 1996).

[T.6] A. Visconti, Introductory differential geometry for physicists (World Scientific, Singapore, 1992).